Introduction to nanophotonics

Part 1 :

(Light properties, Black-body/Density of states, Quantum theory of light polarization)

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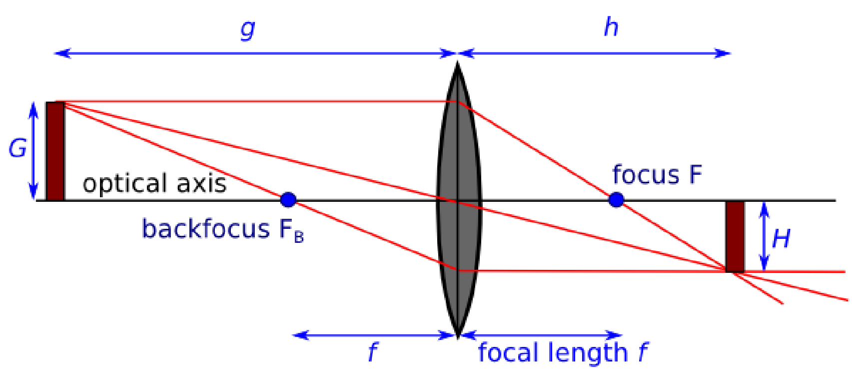
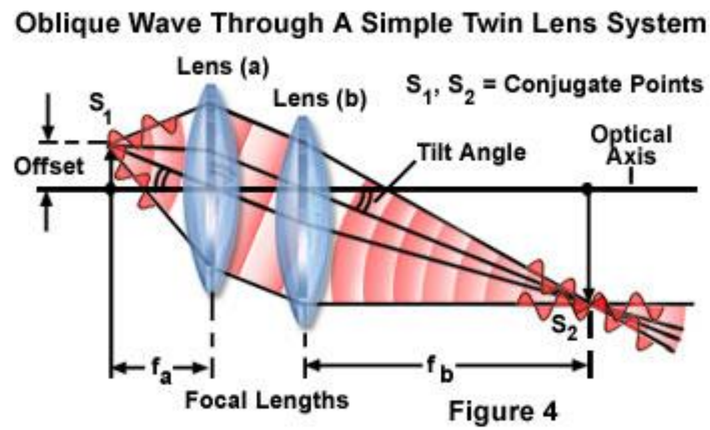
Part 1:

- Properties of light and light propagation
- Density of states and black-body radiation
- Quantum mechanics of light polarization

Geometric optics approximation - Newton:

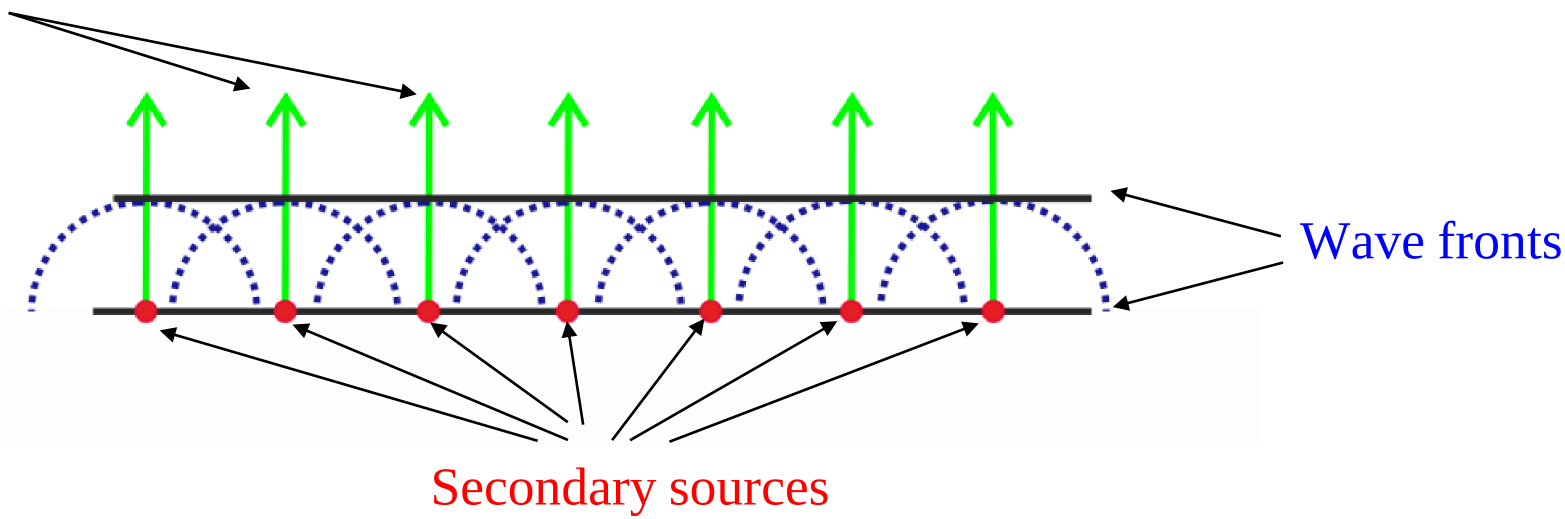
$$\lambda \to 0$$





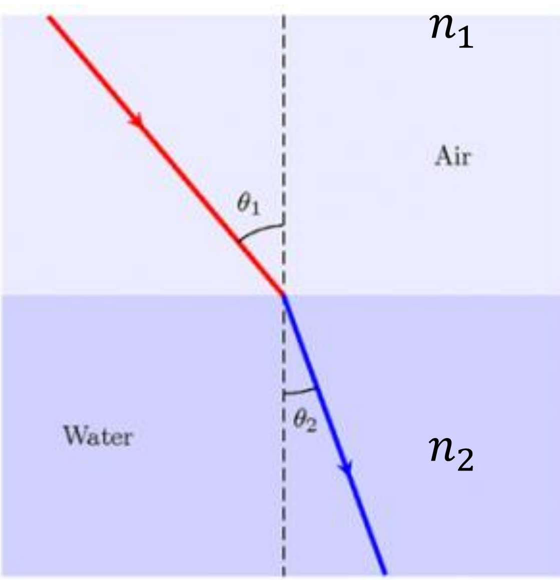
Light as a wave - Huygens principle

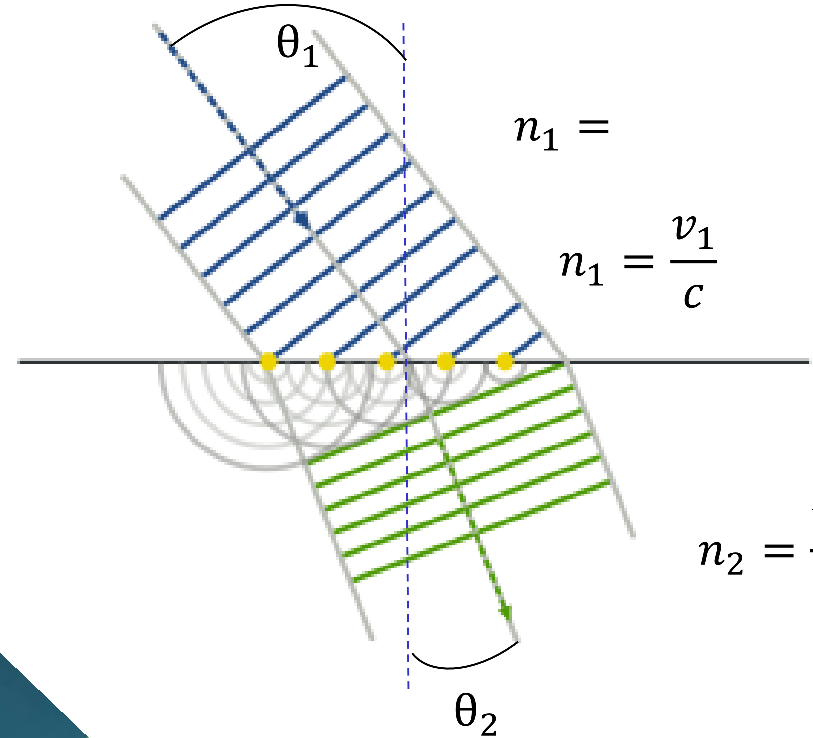
The “propagation” of a wave is can be understood
as the “interference” of secondary sources in the wave fronts



Why do the “secondary” waves only propagate in the forward direction ?
Meta-surfaces give us an answer !

Both Newton's 'particle model' and Huygens's 'wave model' can explain refraction !

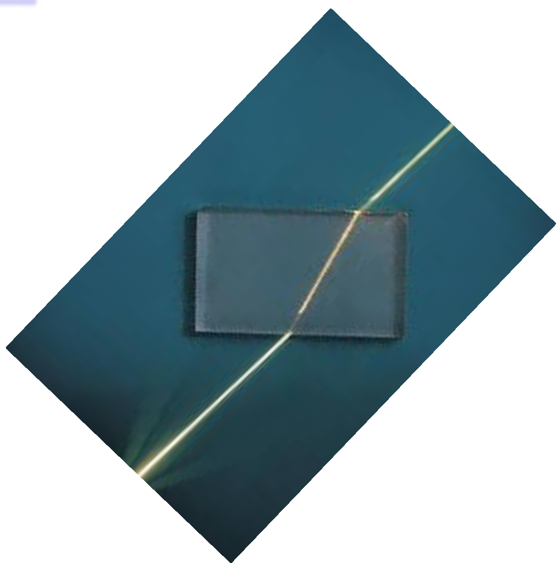




$n_1 =$

$$n_1 = \frac{v_1}{c}$$

$$n_2 = \frac{v_2}{c}$$



Ibn Sahl (983)

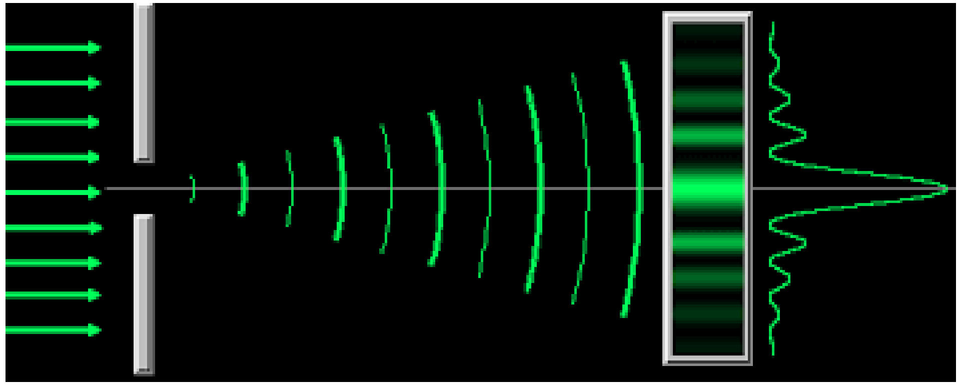
Snell (1621) Descartes (1637)

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

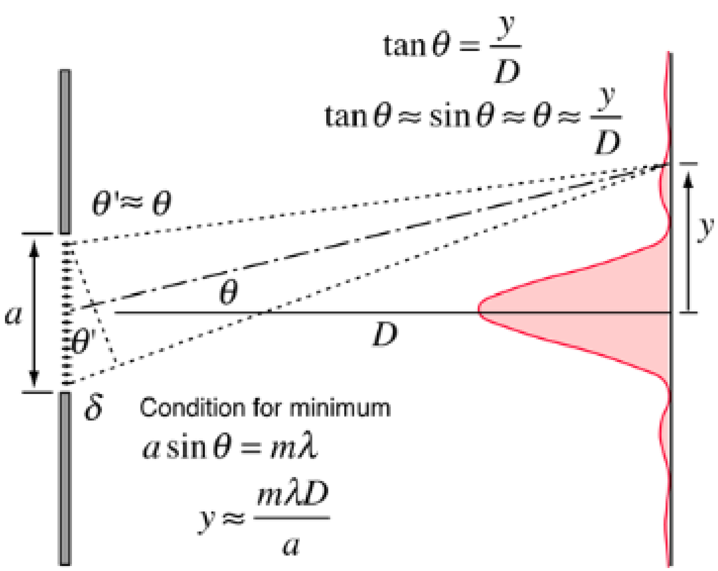
Need a wave theory of light needs to describe diffraction
(Grimaldi ~1665)

Screen

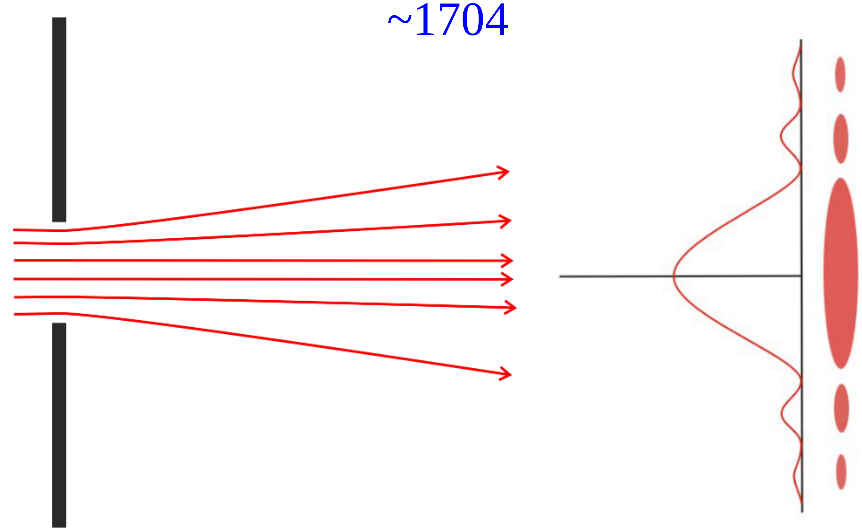
$\lambda \sim a$



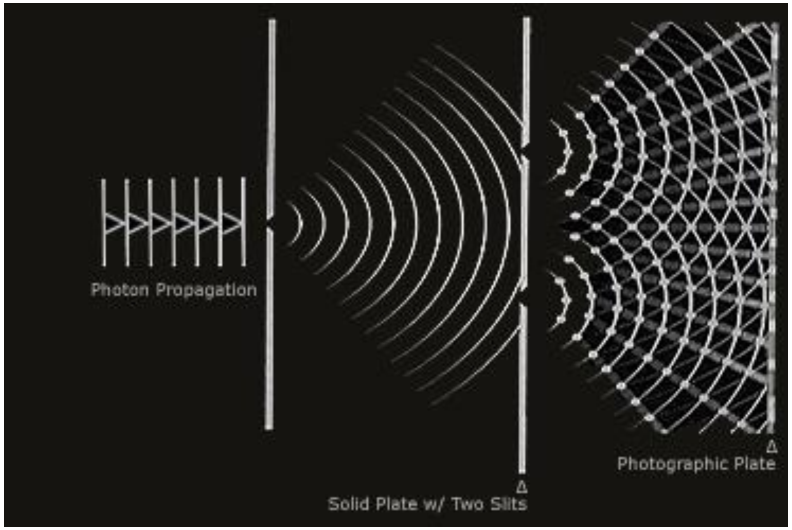
Textbook wave diffraction theory

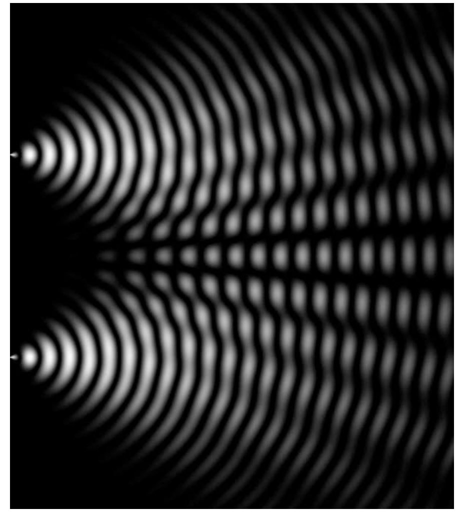


Newton’s particle theory of diffraction ~1704



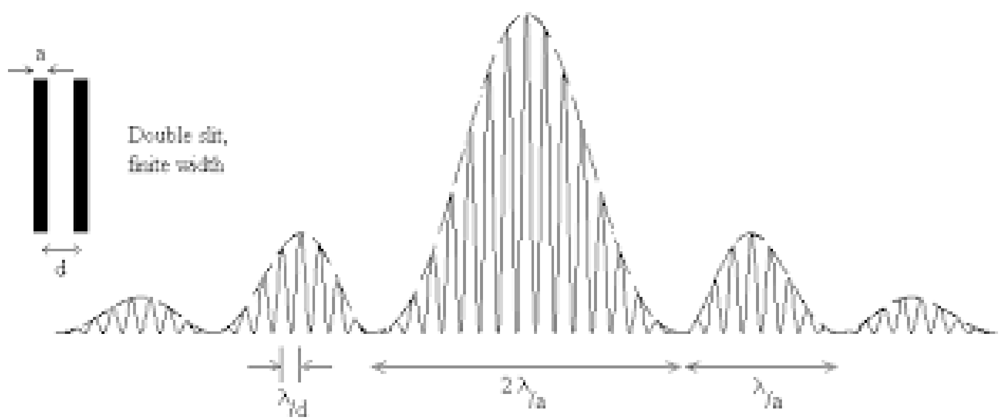
Young's double slit experiment demonstrated the Interference of light waves (1801)

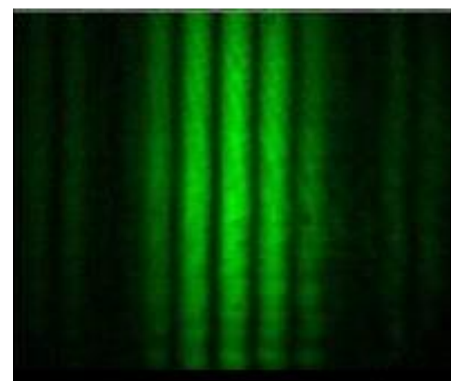




Wave interference

Double slit interference "screen view"





Rayleigh-Sommerfeld diffraction theory ~1900

Fresnel (1819), Green's theorem (1828), Kirchhoff (1883)

'Exact' Rayleigh-Sommerfeld formula
(acoustics, scalar light, quantum mechanics ?)

$$E(x,y,z) = \frac{1}{i\lambda} \iint_{\Sigma} E(X,Y,0) \left(1 + \frac{i}{kr}\right) \frac{\exp(ikr)}{r} F(\theta)\, dX\, dY$$

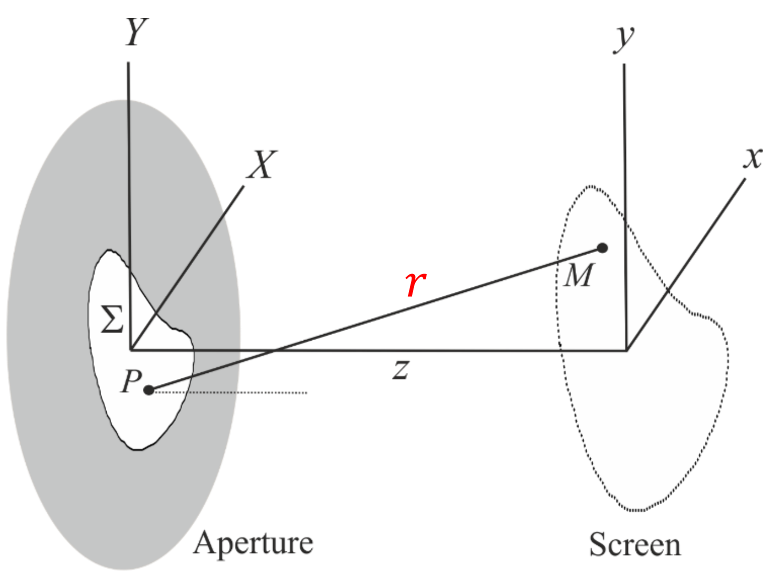
$$r = PM = \sqrt{z^2 + (x-X)^2 + (y-Y)^2}$$

'Obliquity' factor

$$F(\theta) = \cos\theta = \frac{z}{r}$$

wave number

$$k = \frac{2\pi}{\lambda}$$



Green's theorem (1828)

$$\iiint_V (U\Delta G - G\Delta U)d\boldsymbol{r} = \oiint_S \left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right)ds$$

Where $\frac{\partial}{\partial n}$ signifies a partial derivative in the outward normal direction at each point on S

$$\Delta U(\boldsymbol{r}) + k^2 U(\boldsymbol{r}) = 0$$

Green functions :

$$\Delta G(\boldsymbol{r},\boldsymbol{r}') + k^2 G(\boldsymbol{r},\boldsymbol{r}') = -\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')$$

$$G(\boldsymbol{r}_1,\boldsymbol{r}_0) = G(|\boldsymbol{r}_1-\boldsymbol{r}_0|) = G(r_{10}) = \frac{e^{ikr_{10}}}{4\pi r_{10}}$$

$$\frac{\partial G(r_{10})}{\partial n} = \cos(\hat{\boldsymbol{n}},\hat{\boldsymbol{r}}_{10})\left(ik - \frac{1}{r_{10}}\right)\frac{e^{ikr_{10}}}{4\pi r_{10}}$$

Interlude on the scalar Green's function

Homogeneous wave equation : $\Delta U(\boldsymbol{r}) + k^2 U(\boldsymbol{r}) = 0$

Inhomogeneous wave equation : $\Delta \psi(\boldsymbol{r}) + k^2 \psi(\boldsymbol{r}) = -j(\boldsymbol{r})$

Green functions :

$$(\Delta + k^2) G(\boldsymbol{r}, \boldsymbol{r}') = -\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}')$$

$$G(\boldsymbol{r}, \boldsymbol{r}') = (\Delta + k^2)^{-1}$$

$$\psi(\boldsymbol{r}) = \int d\boldsymbol{r}' G(\boldsymbol{r}, \boldsymbol{r}') j(\boldsymbol{r}')$$

$$(\Delta + k^2)\psi(\boldsymbol{r}) = \int d\boldsymbol{r}' (\Delta + k^2) G(\boldsymbol{r}, \boldsymbol{r}') j(\boldsymbol{r}') = -\int d\boldsymbol{r}' \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}') j(\boldsymbol{r}') = -j(\boldsymbol{r})$$

Interlude on the scalar Green's function

How do we know/find that the solution to $(\Delta + k^2)G(\boldsymbol{r}, \boldsymbol{r}') = -\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}')$

$$\text{Is : } G(\boldsymbol{r}_1, \boldsymbol{r}_0) = G(|\boldsymbol{r}_1 - \boldsymbol{r}_0|) = G(r_{10}) = \frac{e^{ikr_{10}}}{4\pi r_{10}}$$

$$\Delta\psi(r,\theta,\phi) = \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2 \sin^2\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\phi^2}\right]$$

$\boldsymbol{r}_0 \to \boldsymbol{0}$

$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_0$

$$G(r,\theta,\phi) = \frac{e^{ikr}}{4\pi r}$$

$$\Rightarrow \quad \Delta G(\boldsymbol{r}) = \frac{1}{r}\frac{\partial^2\left(r\frac{e^{ikr}}{4\pi r}\right)}{\partial r^2} = \frac{1}{4\pi r}\frac{\partial^2\left(e^{ikr}\right)}{\partial r^2} = -\frac{k^2 e^{ikr}}{4\pi r} = -k^2 G(\boldsymbol{r})$$

$$\Rightarrow \quad (\Delta + k^2)\frac{e^{ikr}}{4\pi r} = 0 \qquad r \neq 0$$

Interlude on the scalar Green's function

$$(\Delta + k^2)G(\boldsymbol{r}) = -\delta^{(3)}(\boldsymbol{r}) \qquad G(r) = \frac{e^{ikr}}{4\pi r}$$

Integrate this equation over an infinitely small volume around $\boldsymbol{r} = \boldsymbol{0}$

$$\int_{V\to 0} d\boldsymbol{r}(\Delta + k^2)G(\boldsymbol{r}) = -\int_V d\boldsymbol{r}(\boldsymbol{r})\delta^{(3)} = -1$$

$$\Rightarrow \int_{V\to 0} d\boldsymbol{r}\Delta G(\boldsymbol{r}) = -1 \qquad \Delta G(\boldsymbol{r}) = \nabla\cdot\nabla G(\boldsymbol{r})$$

$$\Rightarrow \int_{V\to 0} d\boldsymbol{r}\nabla\cdot\frac{\boldsymbol{r}}{4\pi r^3} = 1 \ ? \qquad r\to 0 \qquad \nabla G(\boldsymbol{r}) \to -\frac{\boldsymbol{r}}{4\pi r^3}$$

Yes !

$$\int_V d\boldsymbol{r}\nabla\cdot\boldsymbol{A}(\boldsymbol{r}) = \int_S \boldsymbol{A}(\boldsymbol{r})\cdot d\boldsymbol{S} \Rightarrow \int_{V\to 0} d\boldsymbol{r}\nabla\cdot\frac{\boldsymbol{r}}{4\pi r^3} = \int_{S\to 0}\frac{1}{4\pi r^2}\cdot 4\pi r^2 = 1$$

$$F(\theta) = \cos\theta = \frac{z}{r}$$

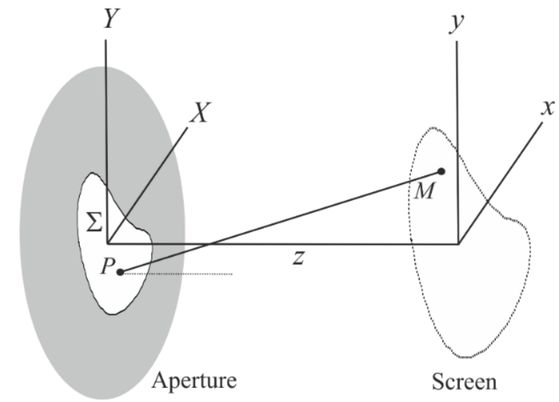
Approximations to Sommerfeld diffraction formula

$$k = \frac{2\pi}{\lambda}$$

$$E(x,y,z) = \frac{1}{i\lambda}\iint_{\Sigma} E(X,Y,0)\left(1+\frac{i}{kr}\right)\frac{\exp(ikr)}{r}F(\theta)dXdY$$

$$\cong \frac{1}{i\lambda}\iint_{\Sigma} E(X,Y,0)\frac{\exp(ikr)}{r}F(\theta)dXdY$$

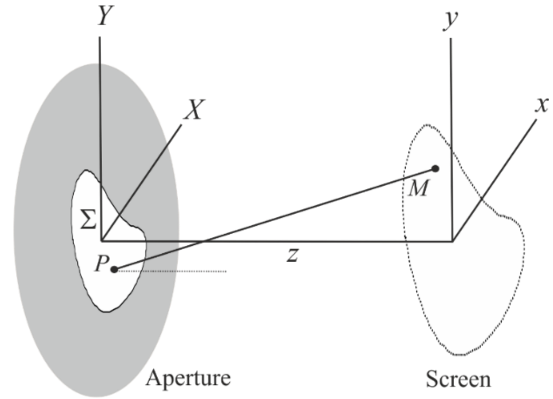
$$r = PM = \sqrt{z^2+(x-X)^2+(y-Y)^2} \cong z\left(1+\frac{[(x-X)^2+(y-Y)^2]}{2z^2}\right)$$



Fresnel-Kirchhoff intermediate-field ‘approximation’ :

$$E(x,y,z) \cong \frac{e^{ikz}}{i\lambda z}\iint_{\Sigma} E(X,Y,0)\exp\left\{\frac{i\pi}{\lambda z}[(x-X)^2+(y-Y)^2]\right\}dXdY$$

Fourier Optics



$$E(x,y,z) \cong \frac{e^{ikz}}{i\lambda z} \iint_{\Sigma} E(X,Y,0) \exp\left\{\frac{ik}{2z}\left[(x-X)^2+(y-Y)^2\right]\right\} dXdY$$

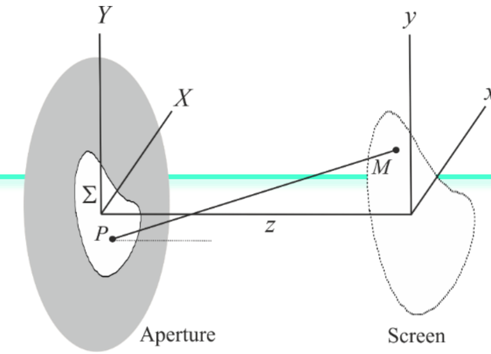
$$E(x,y,z) \cong \frac{e^{ikz}}{i\lambda z} \exp\left[\frac{ik}{2z}(x^2+y^2)\right] \iint_{\Sigma} E(X,Y,0) e^{\frac{ik}{2z}(X^2+Y^2)} e^{\frac{ik}{z}(xX+yY)} dXdY$$

Fresnel-Kirchhoff diffraction :

$$E(x,y,z) \cong C \iint_{\Sigma} E(X,Y) e^{\frac{ik}{2z}(X^2+Y^2)} e^{-\frac{ik}{z}(xX+yY)} dXdY \propto \mathcal{F}\left\{E(X,Y) e^{\frac{ik}{2z}(X^2+Y^2)}\right\}$$

('Parabolic' wavelets)

Fourier Optics in the 'far field'



Fresnel-Kirchhoff diffraction :

$$E(x,y,z) \cong C \iint_{\Sigma} E(X,Y) e^{\frac{ik}{2z}(X^2+Y^2)} e^{-\frac{ik}{z}(xX+yY)} dXdY \propto \mathcal{F}\left\{E(X,Y) e^{\frac{ik}{2z}(X^2+Y^2)}\right\}$$

Fraunhoffer diffraction : $e^{\frac{ik}{2z}(X^2+Y^2)} \Longrightarrow 1 \qquad z \gg X, Y$

$$E(x,y,z) \cong C \iint_{\Sigma} E(X,Y) dXdY \propto \mathcal{F}\{E(X,Y)\}$$

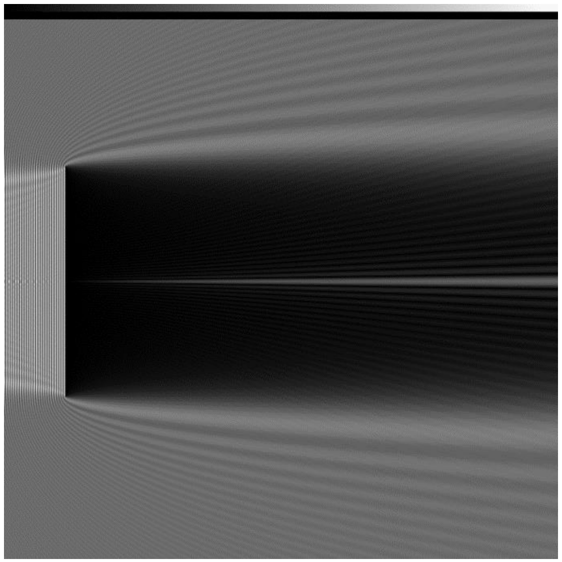
$$\mathcal{F}\{E(X,Y)\} = \iint_{\Sigma} E(X,Y) e^{-\frac{ik}{z}(xX+yY)} dXdY = \iint_{\Sigma} E(X,Y) e^{-ik(X\sin\theta_X + Y\sin\theta_Y)} dXdY$$

$$\sin\theta_X \equiv \frac{X}{z} \qquad \sin\theta_Y \equiv \frac{Y}{z}$$

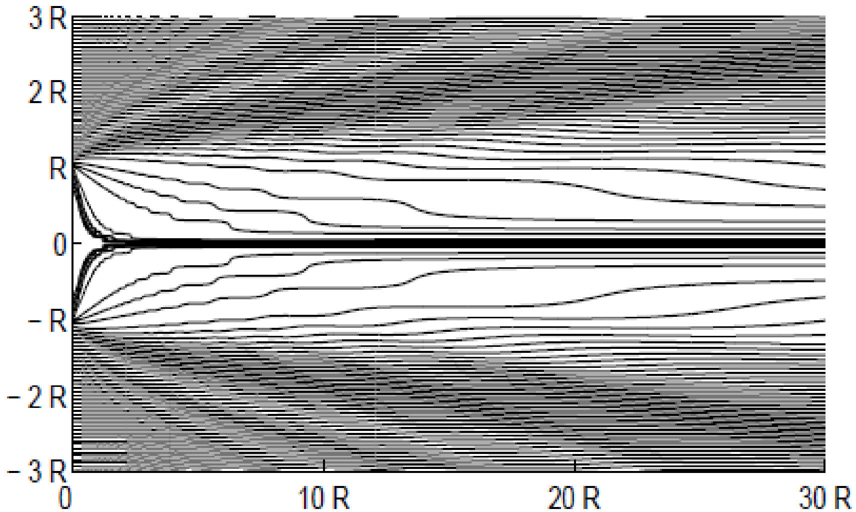
Newton's Particle theory woefully fails to explain the Poisson-Fresnel-Arago "spot"

$$E(x,y,z)=\frac{e^{ikz}}{i\lambda z}\iint_{\Sigma} E(X,Y,0)\exp\left\{\frac{i\pi}{\lambda z}\left[(x-X)^2+(y-Y)^2\right]\right\}dXdY$$

Side views of the 'Poisson' spot 'simulations'



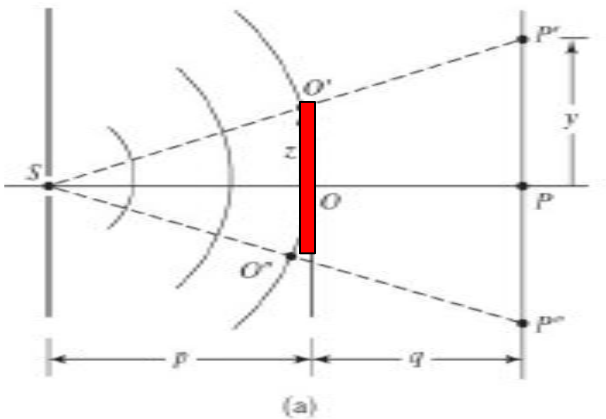
$|E|^2$



$\vec{S}$

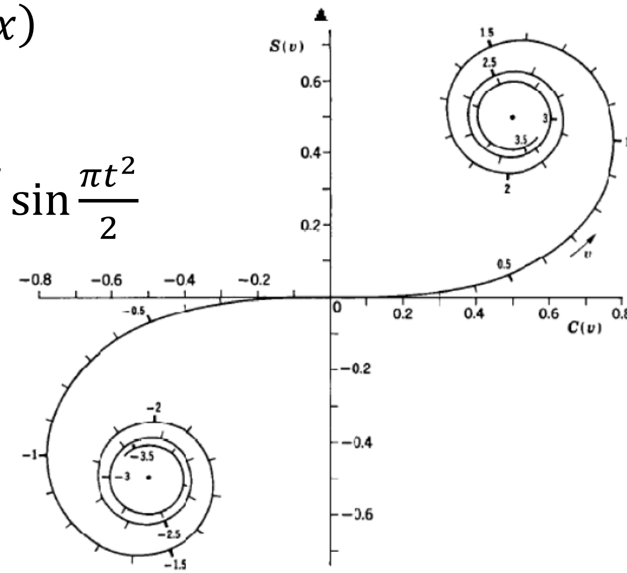
What actually impressed the French academy ?

$$E(x,y,z) = K(z)\iint_{\Sigma} E(X,Y,0)\exp\left\{\frac{i\pi}{\lambda z}[(x-X)^2+(y-Y)^2]\right\} dX dY$$

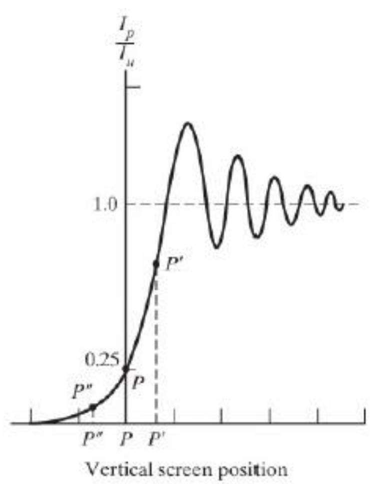


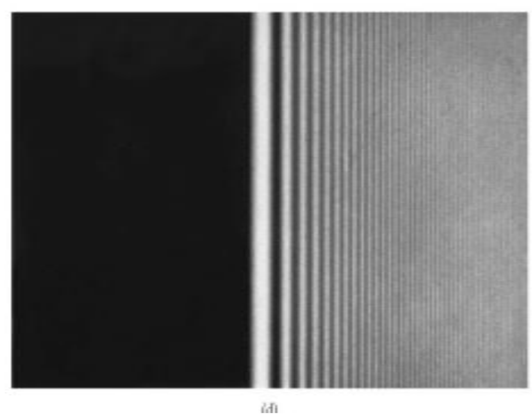
$$I(x) = \int_0^x e^{\frac{i\pi t^2}{2}} dt = C(x) + iS(x)$$

$$C(x) = \int_0^x \cos\frac{\pi t^2}{2} dt \qquad S(x) = \int_0^x \sin\frac{\pi t^2}{2}$$



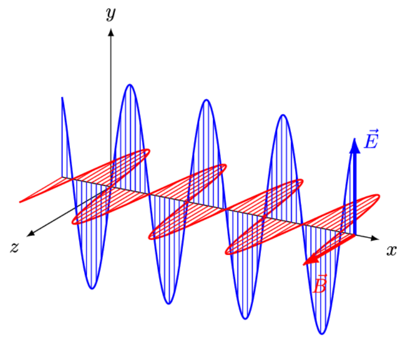
- Precise measurements
- Rigorous mathematics
- Parameter free theory

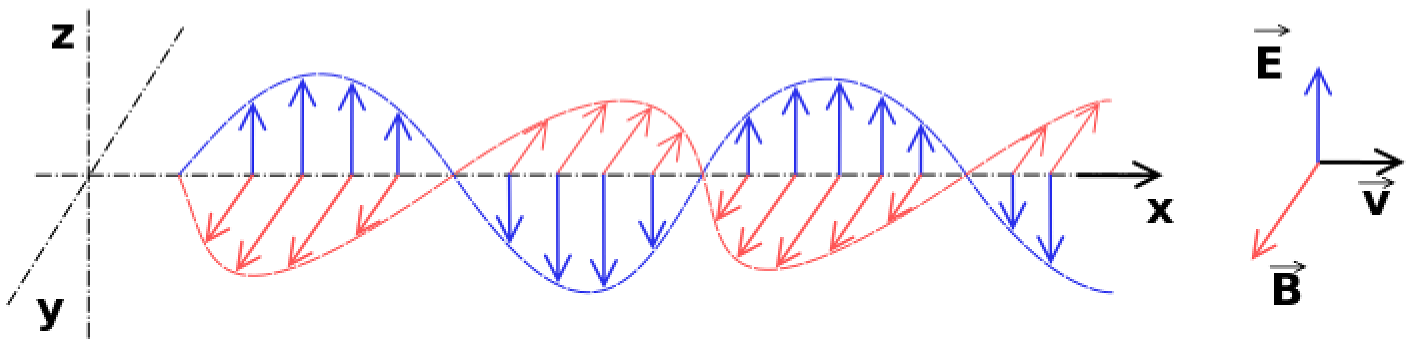




$|\boldsymbol{E}|^2/|\boldsymbol{E}_0|^2$

Since Maxwell's 1865 theory
light is an electromagnetic wave





Explains most of the physics of the Fresnel Institute !
("photonics")

It took ~100 years before one began to seriously explain light phenomena in terms of Maxwell equations (why is that ?)

Maxwell's equations of electromagnetism in free space

Maxwell equations in free space **:**

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

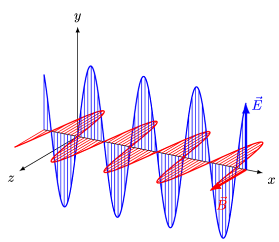
$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = \frac{\partial \boldsymbol{E}}{c^2 \partial t}$$

$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$

$$\left.\begin{aligned} &\nabla \times \nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial^2 \boldsymbol{E}(\boldsymbol{r}, t)}{c^2 \partial t^2} \\ &\nabla \times \nabla \times \boldsymbol{A} \equiv \nabla(\nabla \cdot \boldsymbol{A}) - \Delta \boldsymbol{A} \end{aligned}\right\}$$

$$\Delta \boldsymbol{E}(\boldsymbol{r}, t) - \frac{\partial^2 \boldsymbol{E}(\boldsymbol{r}, t)}{c^2 \partial t^2} = 0$$

$$\Delta \boldsymbol{B}(\boldsymbol{r}, t) - \frac{\partial^2 \boldsymbol{B}(\boldsymbol{r}, t)}{c^2 \partial t^2} = 0$$

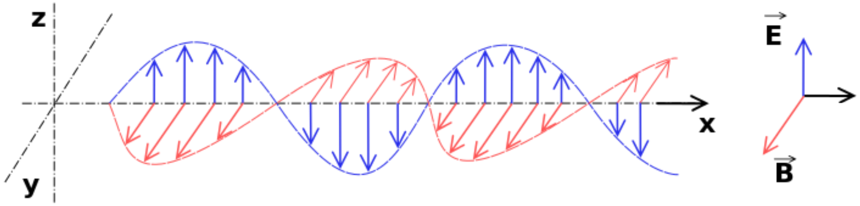


$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathcal{E} \vec{\boldsymbol{e}}_{\boldsymbol{p}} e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$$

$$\omega \equiv 2\pi\nu$$

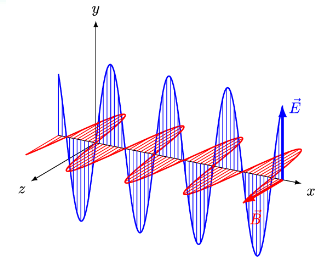
$$k \equiv |\boldsymbol{k}| \equiv \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

Introduction to Quantum Physics with polarization



$\vec{e}_{k_\mathrm{i}}$

Incident `plane' wave *approximation*



Time averaged *Poynting* vector

$$\langle \vec{\mathbf{\Pi}}_\mathrm{inc} \rangle_T = \frac{1}{2} \mathrm{Re}\{\vec{\boldsymbol{E}}^*_\mathrm{inc} \times \vec{\boldsymbol{H}}_\mathrm{inc}\} = \frac{1}{2}\sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}} \left\|\vec{\boldsymbol{E}}_\mathrm{inc}\right\|^2 \vec{\boldsymbol{e}}_{k_\mathrm{i}} = \frac{\epsilon_0 c^2}{2}\sqrt{\frac{\varepsilon_b}{\mu_b}} \left\|\vec{\boldsymbol{E}}_\mathrm{inc}\right\|^2 \vec{\boldsymbol{e}}_{k_\mathrm{i}}$$

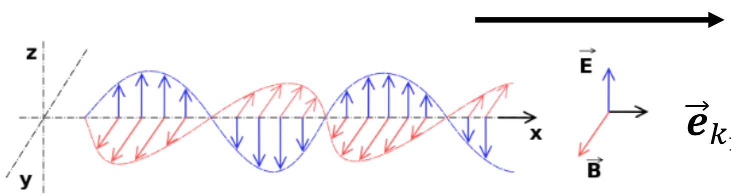
https://www.youtube.com/watch?v=GMmhSext9Q8

$$I = I_R + I_T$$

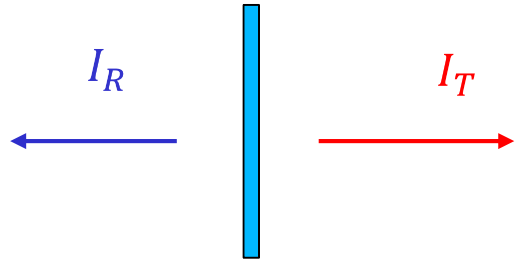
Energy conservation !

$$I \propto \left\|\vec{\mathbf{\Pi}}_\mathrm{inc} \cdot \vec{\boldsymbol{e}}_{k_\mathrm{i}}\right\| \propto \left\|\vec{\boldsymbol{E}}_\mathrm{inc}\right\|^2$$

Perfect *lossless* polarizer



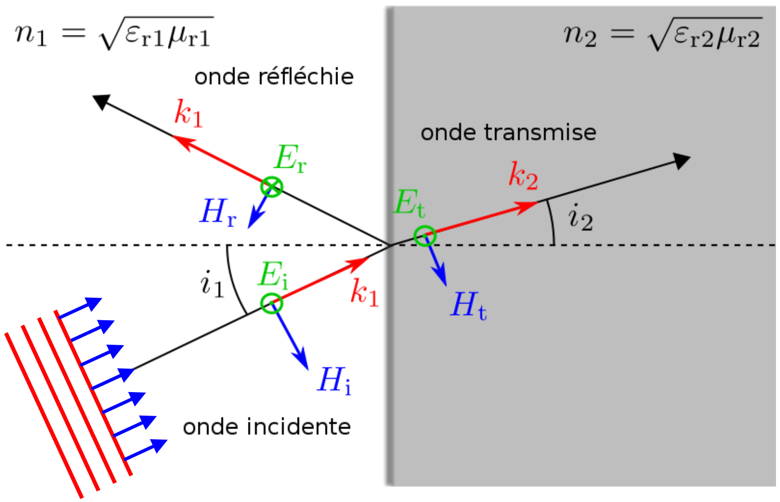
$\vec{e}_{k_\mathrm{i}}$



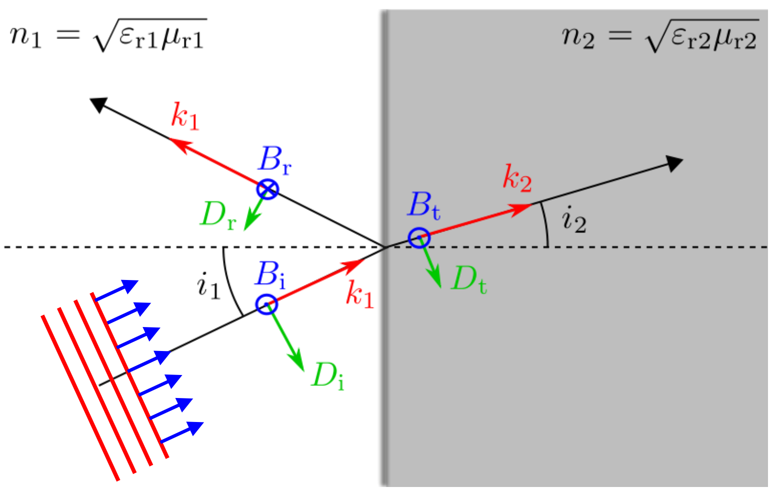
I_R I_T

Wave equations remain indispensable to describe light propagation like Fresnel coefficients

s-polarization



p-polarization



Fresnel coefficients

plane waves, planar interface,

$$r_s = r_\perp = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

$$t_s = t_\perp = \frac{2 n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

$$r_p = r_\parallel = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_1 \cos\theta_t + n_2 \cos\theta_i}$$

$$t_p = t_\parallel = \frac{2 n_1 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i}$$

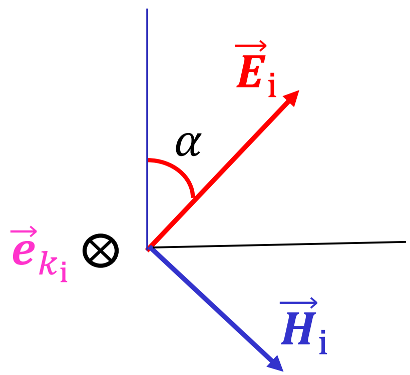
$$n(\omega) = \sqrt{\varepsilon_r(\omega)\mu_r(\omega)}$$

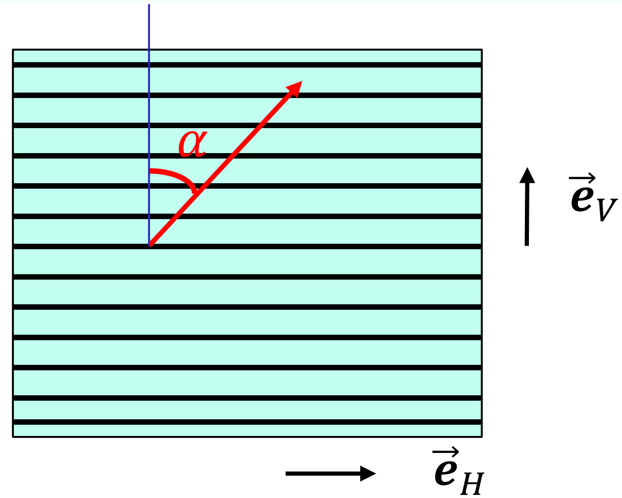
Malus Law : Polarizer

https://www.youtube.com/watch?v=-ZUw1qJOflU

Polarized incident `plane' wave

$$\vec{\Pi}_{\mathrm{i}} = \frac{1}{2}\mathrm{Re}\{\vec{E}_{\mathrm{i}}^{*} \times \vec{H}_{\mathrm{i}}\}$$





$$I = I_R + I_T = \kappa_R I + \kappa_T I$$

Energy conservation !

$$\kappa_R + \kappa_T = 1 \Longrightarrow \begin{cases} \kappa_T = \cos^2\alpha \\ \kappa_R = \sin^2\alpha \end{cases}$$

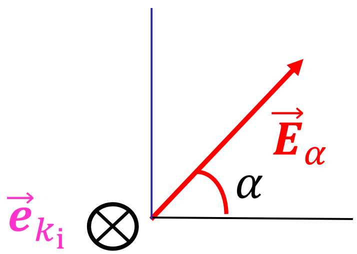
Electric field is a ***vector*** :

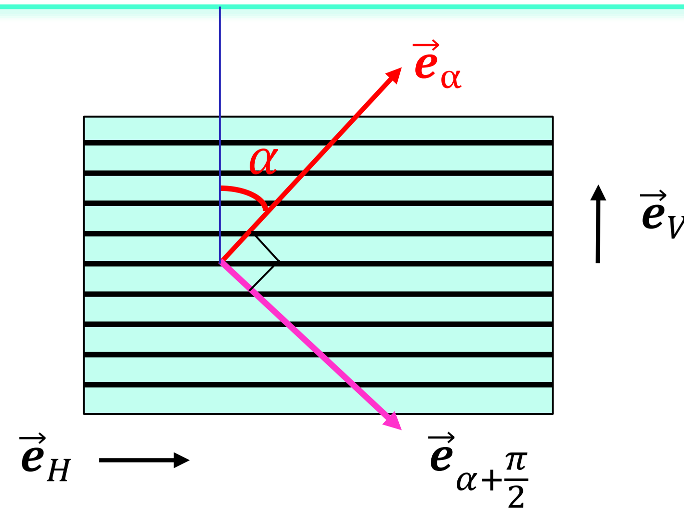
$$\vec{E}_{\mathrm{i}} = \vec{E}_{\alpha} = \mathcal{E}(\cos\alpha\,\vec{e}_V + \sin\alpha\,\vec{e}_H) = \|\vec{E}\|\vec{e}_{\alpha} = \mathcal{E}\vec{e}_{\alpha}$$

$$I \propto \|\vec{E}\|^2 = |\mathcal{E}|^2$$

Basis vectors for electric field polarization

Linearly polarized incident `plane' wave





$$\vec{e}_\alpha = \cos\alpha\,\vec{e}_V + \sin\alpha\,\vec{e}_H$$

$$\vec{E}_\alpha = E\vec{e}_\alpha$$

$$\vec{e}_{\alpha+\frac{\pi}{2}} = \cos\left(\alpha+\frac{\pi}{2}\right)\vec{e}_V + \sin\left(\alpha+\frac{\pi}{2}\right)\vec{e}_H$$

$$= -\sin\alpha\,\vec{e}_V + \cos\alpha\,\vec{e}_H$$

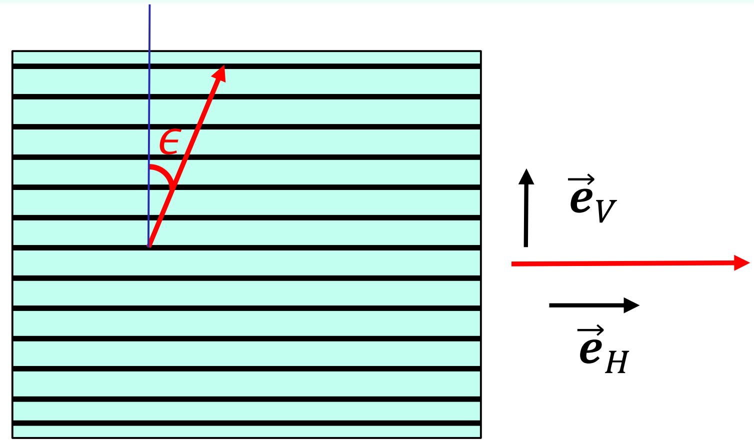
$$\vec{e}_\alpha\cdot\vec{e}_\alpha = 1 \qquad \vec{e}_{\alpha+\frac{\pi}{2}}\cdot\vec{e}_{\alpha+\frac{\pi}{2}} = 1 \qquad \vec{e}_\alpha\cdot\vec{e}_{\alpha+\frac{\pi}{2}} = \vec{e}_{\alpha+\frac{\pi}{2}}\cdot\vec{e}_\alpha = 0$$

Measurement influences propagation
Multiple polarizers

Power transmission and reflection coefficients

$$T_1 \equiv \frac{I_T}{I} = \cos^2 \epsilon$$

$$T_1 \equiv \frac{I_T}{I} = \cos^2 \epsilon$$



$$N = \frac{\pi}{2\epsilon}$$

Polarization $\vec{\boldsymbol{e}}_V$ to $\vec{\boldsymbol{e}}_H$

$$T_N = \cos^2 \epsilon \times \cos^2 \epsilon \ldots \cos^2 \epsilon$$

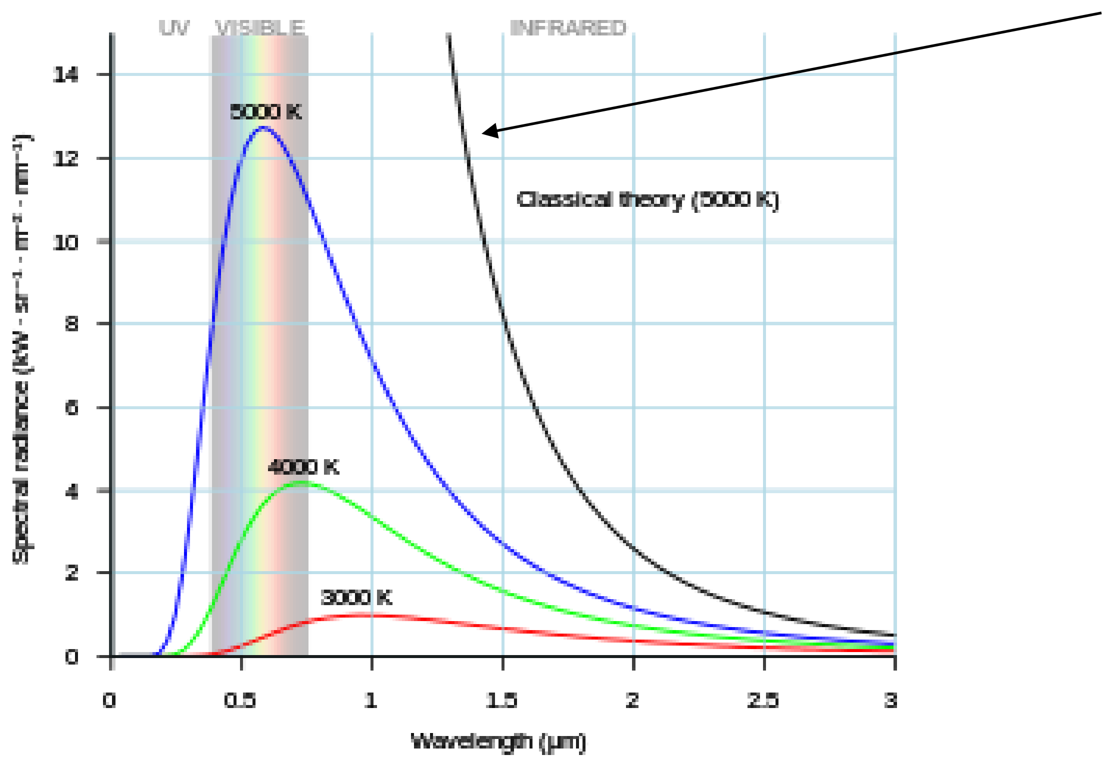
$$\lim_{\epsilon \to 0} (\cos^2 \epsilon)^N = \lim_{\epsilon \to 0} (\cos^2 \epsilon)^{\frac{\pi}{2\epsilon}} \cong \lim_{\epsilon \to 0} \left(1 - \frac{\epsilon^2}{2}\right)^{\frac{\pi}{2\epsilon}} \cong \lim_{\epsilon \to 0} \left(1 - \frac{\pi}{2\epsilon} \frac{\epsilon^2}{2}\right) = 1$$

Quantum theory got started with light!

Black body radiation Planck (1900)

Thermal radiation states, density of states, and Planck's black-body radiation formula

Ultra-violet "catastrophe"



Thermal Radiation : $E_{n_\ell} = n_\ell \hbar\omega_\ell$

Thermal Radiation States of a Single Field Mode (incoherent superposition!)

Density matrix of a mode ℓ :

$$\hat{\rho}_\ell = \sum_{n_\ell=0}^{\infty} p_{n_\ell} |\psi_{n_\ell}\rangle\langle\psi_{n_\ell}| = \sum_{n_\ell=0}^{\infty} p_{n_\ell} |n_\ell\rangle\langle n_\ell|$$

Boltzmann probability distribution

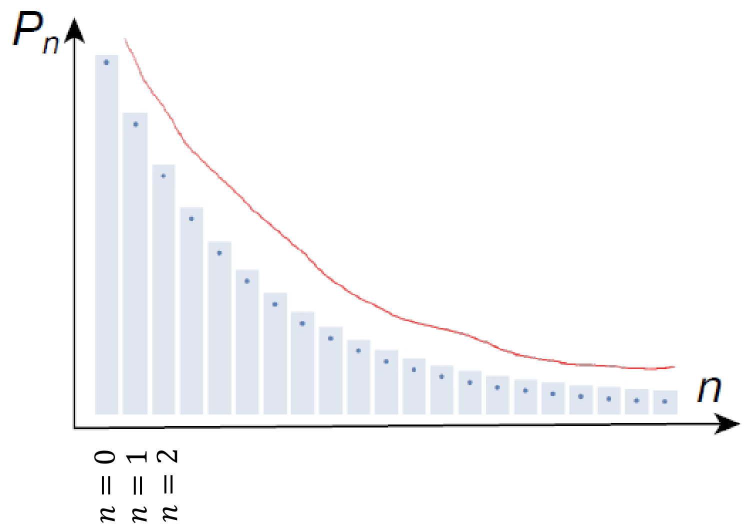
$$p_{n_\ell,\mathrm{Th}} = \frac{e^{-\frac{E_{n_\ell}}{k_B T}}}{Z_\ell}$$

Z_ℓ is chosen such that:

$$x \equiv e^{-\frac{\hbar\omega}{k_B T}}$$

$$\sum_{n=0}^{\infty} p_{n_\ell} = \sum_{n=0}^{\infty} \frac{e^{-\frac{E_{n_\ell}}{k_B T}}}{Z_\ell} = \frac{1}{Z_\ell}\sum_{n=0}^{\infty} x^n = 1$$

$$\Rightarrow \quad Z_\ell = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = \frac{1}{1-e^{-\frac{\hbar\omega}{k_B T}}}$$



Useful properties of density matrices:

$\mathrm{Tr}\{\hat{\rho}\} = 1$

Expectation values : $\langle \hat{A} \rangle = \mathrm{tr}\{\hat{\rho}\hat{A}\} = \mathrm{tr}\{\rho A\}$

Time evolution (von Neumann equation) : $i\hbar \frac{\partial}{\partial t}\hat{\rho} = [\hat{H}, \hat{\rho}]$

Pure states: $\mathrm{tr}\{\rho^2\} = 1$ Ex: $\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\rho^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Mixed states $\mathrm{tr}\{\rho^2\} < 1$ Ex: $\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ $\rho^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

Average thermal mode occupation

Thermal radiation state of a single electromagnetic mode

Attn! : we drop the mode index ℓ

$$\bar{n}_\omega \equiv \langle \hat{n} \rangle = \mathrm{Tr}\left(\hat{n}\hat{\rho}_{n_\ell,\mathrm{Th}}\right) = \sum_{n=0}^{\infty} n p_n = \frac{1}{Z}\sum_{n=0}^{\infty} n e^{-\frac{E_n}{k_B T}} = \frac{1}{Z}\sum_{n=0}^{\infty} n e^{-\frac{n\hbar\omega}{k_B T}}$$

$$E_n = n\hbar\omega$$

$$= \frac{1}{Z}\sum_{n=0}^{\infty} n \left(e^{-\frac{\hbar\omega}{k_B T}}\right)^n = \frac{1}{Z}\sum_{n=0}^{\infty} n x^n$$

$$x \equiv e^{-\frac{\hbar\omega}{k_B T}}$$

$$Z = \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{k_B T}} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$= \frac{1}{Z}\sum_{n=0}^{\infty} x \frac{d}{dx} x^n = \frac{1}{Z} x \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{1}{Z} x \frac{d}{dx}\frac{1}{1-x}$$

$$= \frac{1}{Z}\frac{x}{(1-x)^2} = \frac{x}{1-x} = \frac{1}{\frac{1}{x}-1} = \frac{1}{e^{\frac{\hbar\omega_\ell}{k_B T}}-1}$$

$$\bar{n}_{\omega_\ell} = \frac{1}{e^{\frac{\hbar\omega_\ell}{k_B T}}-1}$$

Bose-Einstein distribution (for photons)

$$\bar{n}_\omega = \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

$\bar{n}_\omega$: average photon number in mode

$$\bar{E}_\omega = \bar{n}_\omega \hbar\omega$$

$\bar{E}_\omega$: Average photon energy

$\hbar\omega$: Single photon energy

Planck radiation formula

$$u(\omega)d\omega = \bar{n}_\omega \hbar\omega \frac{dN}{d\omega} d\omega \frac{1}{V}$$

$u(\omega)d\omega$: Energy density in the frequency interval $\{\omega, \omega + d\omega\}$

$\frac{dN}{d\omega} d\omega$: # of states in the frequency interval $\{\omega, \omega + d\omega\}$

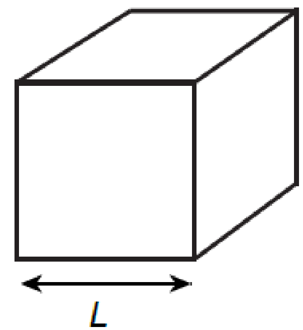
V volume of the `box'

Density of states in a box : $\frac{dN}{d\omega}$

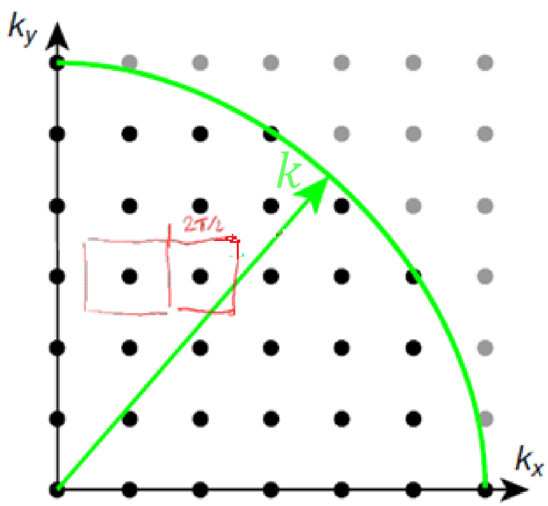
Discretized radiation modes

$$k_x = \frac{2\pi}{L} n_x \qquad k_y = \frac{2\pi}{L} n_y \qquad k_z = \frac{2\pi}{L} n_z$$

$$n_i \in -\infty, \ldots, -1, 0, 1, 2, \ldots \infty$$



$$V = L^3$$



$$N(k) = 2\frac{\frac{4\pi}{3}k^3}{\left(\frac{2\pi}{L}\right)^3} = 2\frac{V}{6\pi^2}k^3$$

2 polarization states per $\boldsymbol{k}$-vector

$$dN = \frac{V}{\pi^2}k^2 dk$$

$$dN = \frac{V}{\pi^2 c^3}\omega^2 d\omega$$

$$k = \frac{\omega}{c} \qquad dk = \frac{d\omega}{c}$$

$$dN = \frac{V}{\pi^2 c^3}\omega^2 d\omega$$

$$\boxed{\frac{dN}{d\omega} = \frac{V}{\pi^2 c^3}\omega^2}$$

Planck radiation formula :

Density of states : $\frac{dN}{d\omega} = \frac{V\omega^2}{\pi^2 c^3}$

$$u(\omega)d\omega = \bar{n}_\omega \hbar\omega \frac{dN}{d\omega}\frac{1}{V}d\omega$$

Density of states per unit volume : $\frac{dn}{d\omega} \equiv \frac{1}{V}\frac{dN}{d\omega} = \frac{\omega^2}{\pi^2 c^3}$

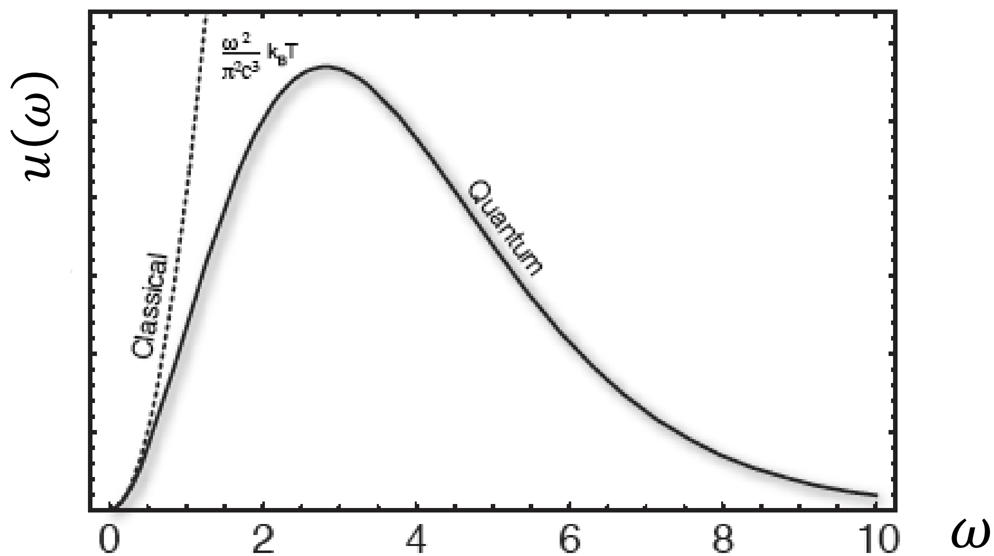
$$u(\omega) = \frac{1}{e^{\hbar\omega/k_B T} - 1}\hbar\omega\frac{\omega^2}{\pi^2 c^3}$$

$$\bar{n}_\omega = \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/k_B T} - 1}$$

Planck radiation formula vs classical prediction :

$$u(\omega) = \frac{1}{e^{\hbar\omega/k_B T} - 1}\hbar\omega\frac{\omega^2}{\pi^2 c^3} = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/k_B T} - 1}$$



$\hbar\omega/k_B T \to 0$

Classical prediction for the density of states $u(\omega) \cong \frac{\omega^2}{\pi^2 c^3} k_B T$

Photon fluctuations in black-body radiation :

$$\langle \hat{n}^2 \rangle = \mathrm{Tr}\left(\hat{n}^2 \hat{\rho}_{n_{\ell,\mathrm{Th}}}\right) = \sum_{n=0}^{\infty} n^2 p_n = \frac{1}{Z}\sum_{n=0}^{\infty} n^2 e^{-\frac{E_n}{k_B T}} = \frac{1}{Z}\sum_{n=0}^{\infty} n^2 x^n$$

$$= \frac{1}{Z}\sum_{n=0}^{\infty} n^2 x^n = \frac{1}{Z} x \frac{d}{dx} x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \frac{1}{Z} x \frac{d}{dx} x \frac{d}{dx} \frac{1}{1-x}$$

$$= \frac{1}{Z} x \frac{d}{dx} x(1-x)^{-2} = \frac{1}{Z}\left[x(1-x)^{-2} + 2x^2(1-x)^{-3}\right]$$

$$= x(1-x)^{-1} + 2x^2(1-x)^{-2}$$

$$= \frac{1}{\frac{1}{x}-1} + \frac{2}{\left(\frac{1}{x}-1\right)^2} = \langle \hat{n} \rangle + 2\langle \hat{n} \rangle^2$$

$$x \equiv e^{-\frac{\hbar\omega}{k_B T}}$$

$$\langle \hat{n}^2 \rangle = \langle \hat{n} \rangle + 2\langle \hat{n} \rangle^2 = \bar{n}_\omega + 2\bar{n}_\omega^2$$

Photon fluctuations in black-body radiation :

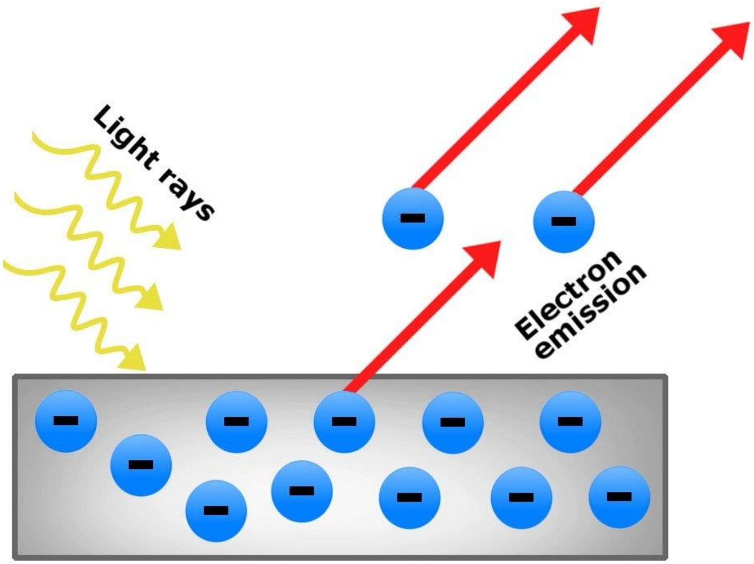
$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = \bar{n}_\omega + \bar{n}_\omega^2$$

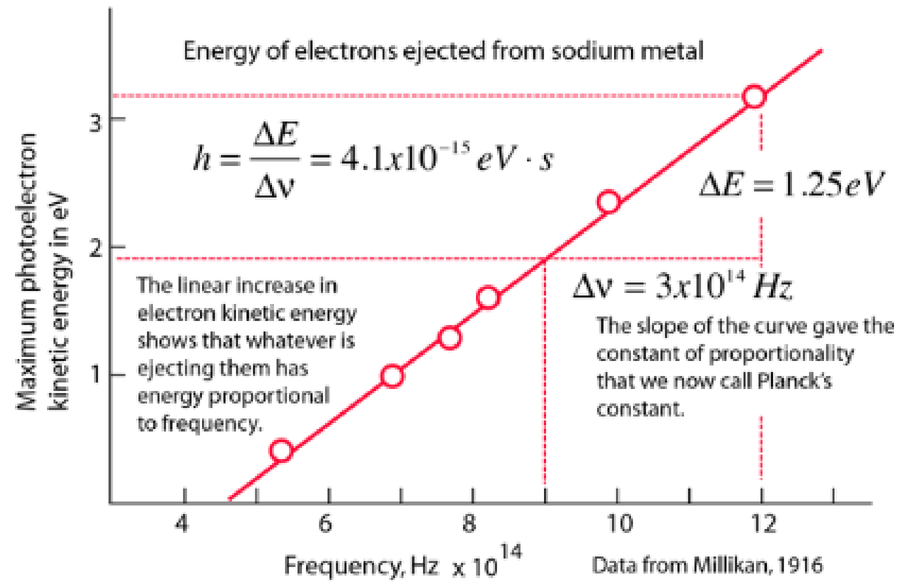
Particle like fluctuations

`Wave like’ fluctuations

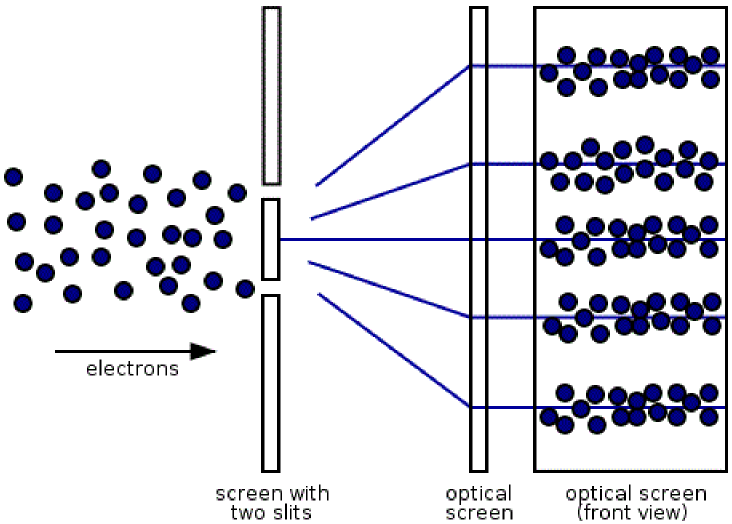
Light became a particle ? “Lichtquanta”

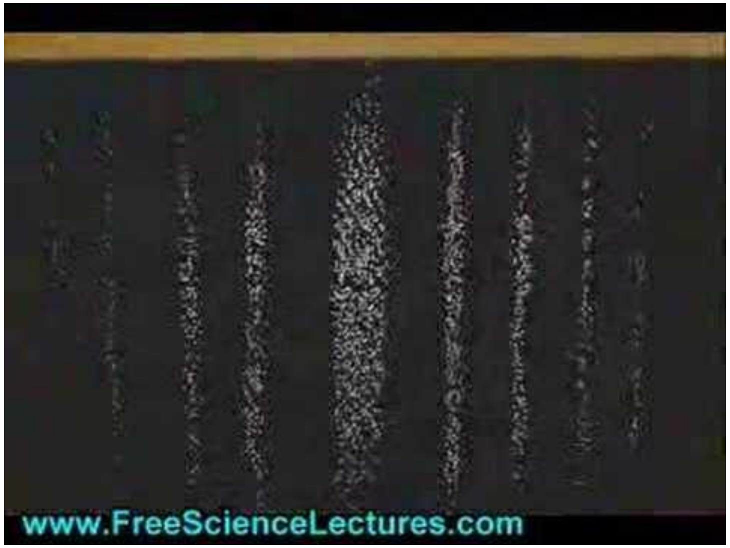
Photo-electric Effect (Einstein 1905)





Quantum mechanics ?
Particles or waves



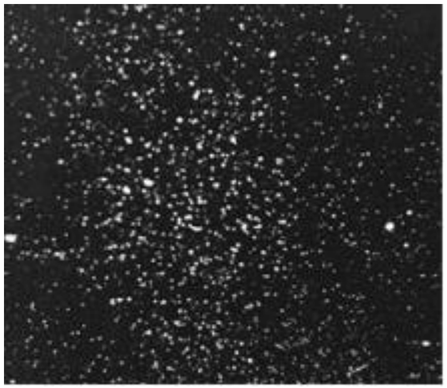


Quantum mechanics played a key role in the technological developments of the 20th century

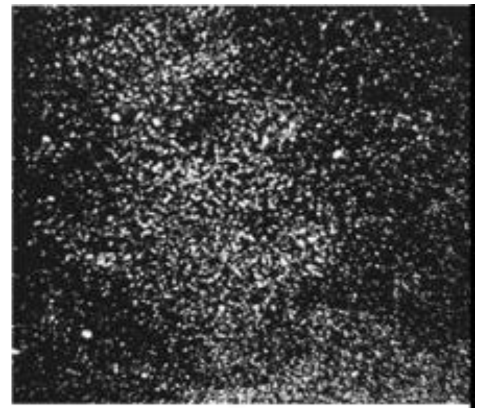
But do photons truly exist ?

(The semi-classical picture of light can explain blackbody radiation, photo-electric effect, stimulated emission (lasers), ultra-fast photography, …)

3×10^3 photons



1.2×10^4 photons



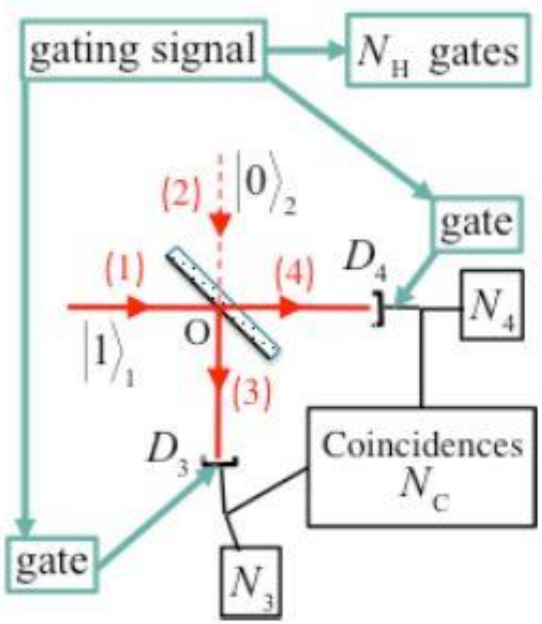
9.3×10^4 photons

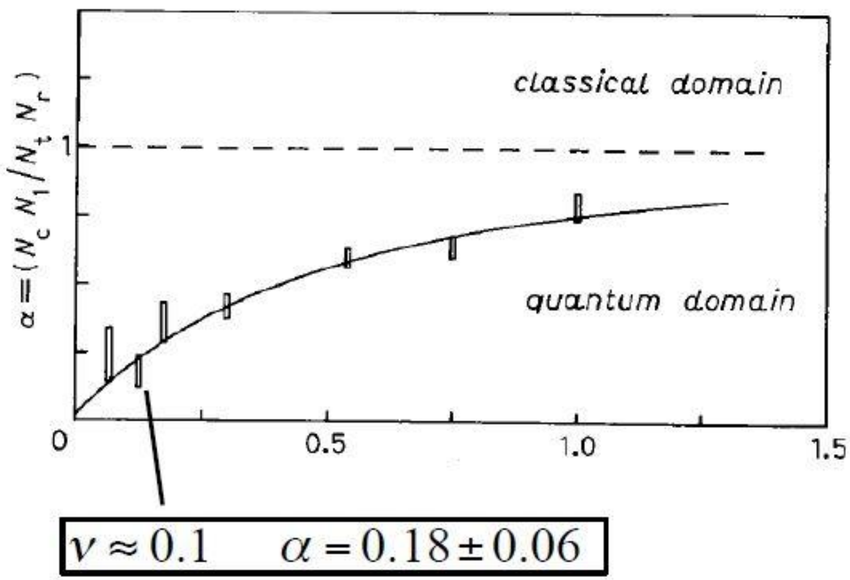
2.8×10^7 photons

Since the 1980's we know that light obeys quantum mechanical laws!
1-photon sources
(Wave particle duality, entanglement, Bell inequalities, quantum cryptography,…)

True 'Quantum Optics' often requires 1-photon (2-photon) sources, and low temperatures !

P. Grangier G. Roger and A. Aspect
EUROPHYSICS LETTERS
Europhys. Lett., **1** (4), pp. 173-179 (1986)

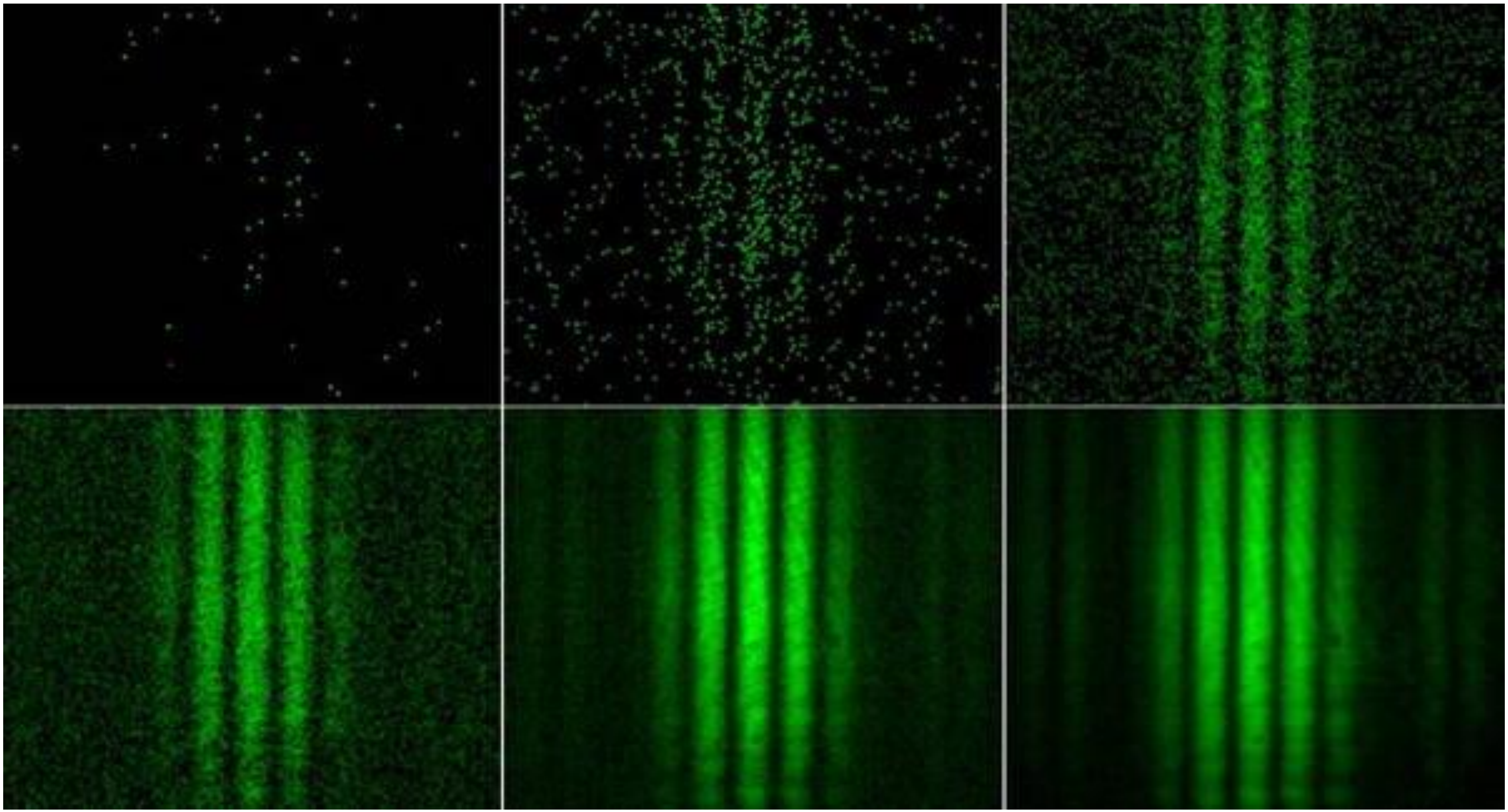




Is this the ***second*** quantum revolution?

French version : https://www.youtube.com/watch?v=_kGqkxQo-Tw
English version : https://www.youtube.com/watch?v=RSXpeDgqUO4
Alain Aspect : https://www.coursera.org/learn/quantum-optics-single-photon

Interference patterns exist even with true 1-photon wave-packets



Fundamental postulate of quantum mechanics

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + c_3|\psi_3\rangle + \ldots = \sum_n c_n|\psi_n\rangle$$

$$\langle\psi| = c_1^*\langle\psi_1| + c_2^*\langle\psi_2| + c_3^*\langle\psi_3| + \ldots = \sum_n c_n^*\langle\psi_n|$$

c_n are **complex-valued** coefficients

$$\langle\psi_m|\psi_n\rangle = \delta_{n,m}$$

$$\langle\psi|\psi\rangle = 1 \Rightarrow \sum_{n=0} |c_n|^2 = 1$$

A nice Youtube video explaining quantum mechanics in the framework of light polarization

https://www.youtube.com/watch?v=-ZUw1qJOflU

Describing 1 photon

Dirac `bra' - `ket' notation $\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle^*$

In quantum physics states of a system are described by superpositions of abstract `vectors': $|\psi\rangle$

`Kets' :

$$\vec{\boldsymbol{e}}_V \rightarrow |V\rangle = \binom{1}{0} \qquad \vec{\boldsymbol{e}}_H \rightarrow |H\rangle = \binom{0}{1}$$

`Bras' :

$$\langle H| = (0,1) \qquad \langle V| = (1,0)$$

$$\vec{\boldsymbol{e}}_H \cdot \vec{\boldsymbol{e}}_H = 1 \rightarrow \langle H|H\rangle = (0,1)\binom{0}{1} = 1$$

$$\vec{\boldsymbol{e}}_V \cdot \vec{\boldsymbol{e}}_V = 1 \rightarrow \langle V|V\rangle = (1,0)\binom{1}{0} = 1$$

$$\vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_{\alpha+\frac{\pi}{2}} = \vec{\boldsymbol{e}}_{\alpha+\frac{\pi}{2}} \cdot \vec{\boldsymbol{e}}_\alpha = 0 \rightarrow \langle V|H\rangle = (1,0)\binom{0}{1} = \langle H|V\rangle = (0,1)\binom{1}{0} = 0$$

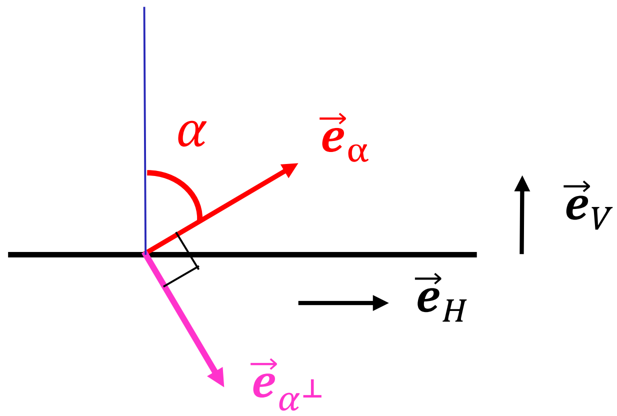
Complete basis of a 2-state system :

$$|V\rangle\langle V| + |H\rangle\langle H| = \binom{1}{0}(1,0) + \binom{0}{1}(0,1) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbb{I}$$

Alternative basis sets for describing 1 photon states

$$|\alpha\rangle = \cos\alpha\big|V\rangle + \sin\alpha\,|H\rangle = \begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}$$

$$|\alpha^{\perp}\rangle = \left|\alpha + \frac{\pi}{2}\right\rangle = -\sin\alpha\big|V\rangle + \cos\alpha\,|H\rangle = \begin{pmatrix}-\sin\alpha \\ \cos\alpha\end{pmatrix}$$



Alternative basis sets

$$\langle\alpha|\alpha\rangle = (\cos\alpha\,,\sin\alpha)\begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix} = \cos^2\alpha + \sin^2\alpha = 1$$

$$\langle\alpha^{\perp}|\alpha^{\perp}\rangle = \left\langle\alpha + \frac{\pi}{2}\middle|\alpha + \frac{\pi}{2}\right\rangle = 1$$

$$\langle\alpha|\alpha^{\perp}\rangle = (\cos\alpha\,,\sin\alpha)\begin{pmatrix}-\sin\alpha \\ \cos\alpha\end{pmatrix} = 0$$

$$\langle\alpha^{\perp}|\alpha\rangle = (-\sin\alpha\,,\cos\alpha)\begin{pmatrix}-\sin\alpha \\ \cos\alpha\end{pmatrix} = 0$$

Complete basis of a 2-state system :

$$|\alpha\rangle\langle\alpha| + |\alpha^{\perp}\rangle\langle\alpha^{\perp}| = \begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}(\cos\alpha\,,\sin\alpha) + \begin{pmatrix}-\sin\alpha \\ \cos\alpha\end{pmatrix}(-\sin\alpha\,,\cos\alpha) = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} = \mathbb{I}$$

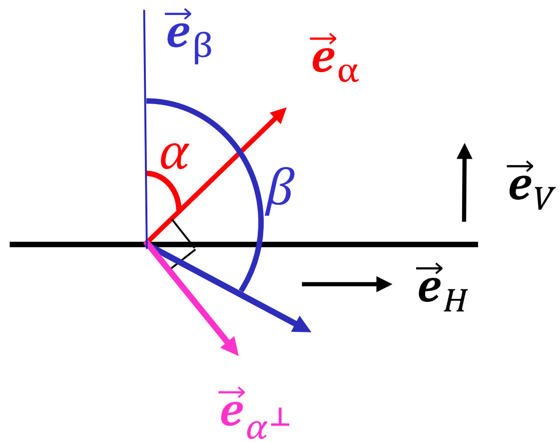
Born’s rule for single particle wave functions

$$P(\text{finding } \psi_1 \text{ given state } \psi_2) = |\langle \psi_1 | \psi_2 \rangle|^2$$

$$|\alpha\rangle = \cos\alpha|V\rangle + \sin\alpha\,|H\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$P(\text{finding } V \text{ given state } \alpha) = |\langle V|\alpha\rangle|^2 = \cos^2\alpha$$

$$P(\text{finding } H \text{ given state } \alpha) = |\langle H|\alpha\rangle|^2 = \sin^2\alpha$$



$$|\beta\rangle = \cos\beta|V\rangle + \sin\beta\,|H\rangle = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$$

$$P(\alpha|\beta) = |\langle\alpha|\beta\rangle|^2 = \left|(\cos\alpha\,, \sin\alpha)\begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}\right|^2 = |\cos\alpha\cos\beta + \sin\alpha\sin\beta|^2 = \cos^2(\beta-\alpha)$$

$$P(\alpha^\perp|\beta) = |\langle\alpha^\perp|\beta\rangle|^2 = \left|(-\sin\alpha\,, \cos\alpha)\begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}\right|^2 = |-\sin\alpha\cos\beta + \cos\alpha\sin\beta|^2 = \sin^2(\beta-\alpha)$$

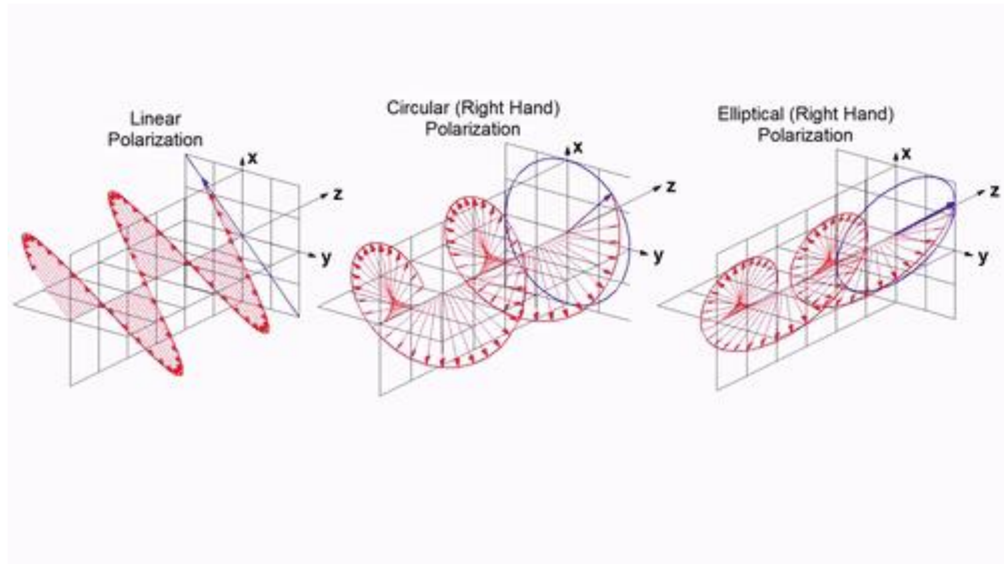
Superposition with complex coefficients (circular/elliptical polarization)

$$|+\rangle = \frac{1}{\sqrt{2}}(|V\rangle + i|H\rangle) = \frac{1}{\sqrt{2}}\binom{1}{i}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|V\rangle - i|H\rangle) = \frac{1}{\sqrt{2}}\binom{1}{-i}$$

$$\langle +| = \frac{1}{\sqrt{2}}(\langle V| - i\langle H|) = \frac{1}{\sqrt{2}}(1 \quad , \quad -i)$$

$$\langle -| = \frac{1}{\sqrt{2}}(\langle V| + i\langle H|) = \frac{1}{\sqrt{2}}(1 \quad , \quad i)$$



Complete basis of a 2-state system :

$$|+\rangle\langle +| + |-\rangle\langle -| = \frac{1}{2}\binom{1}{i}(1, -i) + \frac{1}{2}\binom{1}{-i}(1, i) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$$

Two-particle system (Tensor product)

Tensor product :

$$|V\rangle \otimes |V\rangle = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}1\\0\end{pmatrix}\\0\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \equiv |VV\rangle$$

$$|V\rangle \otimes |H\rangle = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\0\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |H\rangle \otimes |V\rangle = \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |H\rangle \otimes |H\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Arbitrary two-particle state vector :

$$|\psi\rangle = a|VV\rangle + b|VH\rangle + c\big|HV\rangle + d|HH\rangle$$

Normalization :

$$\langle\psi|\psi\rangle = 1 \Rightarrow |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$$

Product states and Entangled states

Product state : $|\Psi_2\rangle = a|HH\rangle + b\big|HV\rangle = |H\rangle(C_\alpha|H\rangle + S_\alpha|V\rangle) = |H\rangle|\alpha\rangle$

$C_\alpha = \cos\alpha \ , \ S_\alpha = \sin\alpha \ ,$

Entangled state : $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$

Prove that $|\Phi^+\rangle$ an Entangled state:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle) \overset{?}{\rightarrow} \left(C_\alpha\big|H\rangle + S_\alpha|V\rangle\right)\left(C_\beta\,\big|H\rangle + S_\beta|V\rangle\right)$$
$$= C_\alpha C_\beta|HH\rangle + C_\alpha S_\beta|HV\rangle + S_\alpha C_\beta|VH\rangle + S_\alpha S_\beta|VV\rangle$$

$$C_\alpha C_\beta = S_\alpha S_\beta = \frac{1}{\sqrt{2}} \qquad\qquad C_\alpha S_\beta = S_\alpha C_\beta = 0$$

Entangled ? Not entangled ?
Normalized ?

$$|\Psi_1\rangle = \frac{1}{2}(|HH\rangle + |HV\rangle + |VH\rangle + |VV\rangle)$$

$$|\Psi_2\rangle = \tfrac{1}{2}(|HH\rangle + |HV\rangle + |VH\rangle - |VV\rangle)$$

$$|\Psi_4\rangle = \cos\theta\,|HH\rangle + \sin\theta\,|VV\rangle$$

$$|\Psi_3\rangle = \frac{1}{2}|HH\rangle + \frac{\sqrt{3}}{2\sqrt{2}}(|VH\rangle + |VV\rangle)$$

Entangled ? Not entangled ? Normalized ?

Solution

$$|\Psi_1\rangle = \frac{1}{2}(|HH\rangle + |HV\rangle + |VH\rangle + |VV\rangle) = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \otimes \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$$ not entangled

$$|\Psi_2\rangle = \frac{1}{2}(|HH\rangle + |HV\rangle + |VH\rangle - |VV\rangle)$$ entangled

$$|\Psi_3\rangle = \frac{1}{2}|HH\rangle + \frac{\sqrt{3}}{2\sqrt{2}}(|VH\rangle + |VV\rangle)$$ entangled

$|\Psi_4\rangle = \cos\theta\,|HH\rangle + \sin\theta\,|VV\rangle$, entangled except when $\theta = 0, \frac{\pi}{2}, \pi,$

$$\langle\Psi_1|\Psi_1\rangle = \langle\Psi_2|\Psi_2\rangle = \langle\Psi_3|\Psi_3\rangle = \langle\Psi_4|\Psi_4\rangle = 1$$

Entanglement doesn't depend on the basis

Show

$$\frac{1}{\sqrt{2}}(|\alpha\alpha\rangle + |\alpha^{\perp}\alpha^{\perp}\rangle) = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$$

Entanglement doesn't depend on the basis

Show

$$\frac{1}{\sqrt{2}}(|\alpha\alpha\rangle + |\alpha^{\perp}\alpha^{\perp}\rangle) = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$$

$$\frac{1}{\sqrt{2}}(|\alpha\alpha\rangle + |\alpha^{\perp}\alpha^{\perp}\rangle) = \left[\frac{1}{\sqrt{2}}(C|H\rangle + S|V\rangle)\frac{1}{\sqrt{2}}(C|H\rangle + S|V\rangle) + \frac{1}{\sqrt{2}}(S|H\rangle - C|V\rangle)\frac{1}{\sqrt{2}}(S|H\rangle - C|V\rangle)\right]$$

$$\frac{1}{\sqrt{2}}(|\alpha\alpha\rangle + |\alpha^{\perp}\alpha^{\perp}\rangle) = \frac{1}{\sqrt{2}}[(C^2 + S^2)|HH\rangle + (CS - SC)|HV\rangle + (SC - CS)|VH\rangle + (S^2 + C^2)|VV\rangle]$$

$$= \frac{1}{\sqrt{2}}[|HH\rangle + |VV\rangle]$$

Bell states of maximal entanglement

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

For a Bell state (of a 2-particle system), knowledge of the state of 1 particle perfectly determines the state of the other

Subject of the 2022 Nobel Prize in physics, basis for the most popular form of quantum cryptography.